

TANGENT 2-FIELDS ON EVEN-DIMENSIONAL NONORIENTABLE MANIFOLDS

BY

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ABSTRACT. This paper uses the Postnikov decomposition of a nonsimple fibration to describe the obstructions to a tangent 2-field on an even-dimensional nonorientable manifold.

Our purpose in this article is to prove the following theorem.

MAIN THEOREM. *Let M be a closed, nonorientable manifold of dimension m , where m is even and $m > 2$. Let $\mathcal{X}(M)$ in $H^m(M; Z_{w_1(M)})$ and $\mathcal{W}_{m-1}(M)$ in $H^{m-1}(M; Z_{w_1(M)})$ denote the twisted Euler class of M and the $(m - 1)$ Stiefel-Whitney class of M , respectively. Then M has a 2-field if, and only if, $\mathcal{X}(M) = 0$ and $\mathcal{W}_{m-1}(M) = 0$.*

The vector field problem has a long history, and in the special case of 2-fields a great deal is already known. In order to put this result in perspective it will be helpful to give a brief statement of the known results. For details and an extensive bibliography the reader is referred to the excellent survey paper by Emery Thomas [10].

Thomas [10], Frank [3], and Atiyah [1] have described completely the necessary and sufficient conditions for a closed orientable manifold to have a 2-field. Suppose M is such a manifold of dimension m . Let $w_{m-1}(M)$ in $H^{m-1}(M; Z_2)$ be the $(m - 1)$ Stiefel-Whitney class, $X(M)$ the Euler class, $k(M)$ the real Kervaire semicharacteristic, and $\sigma(M)$ the signature of M . Their results may then be summarized as follows.

Dimension of $M \bmod 4$	Necessary and sufficient conditions for a 2-field on M
3	M always has a 2-field
2	$X(M) = 0$
1	$w_{m-1}(M) = 0$ and $k(M) = 0$
0	$w_{m-1}(M) = 0$, $X(M) = 0$, and $\sigma(M) \equiv 0 \pmod{4}$

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In addition, Atiyah and Dupont [2] have given necessary and sufficient conditions for a 2-field on a nonorientable $4k + 1$ manifold which satisfies $w_1^2(M) = 0$. Their result is somewhat complicated, and we will not attempt to state it here.

A summary of nonsimple Postnikov theory. Postnikov theory in the nonsimple case has been developed extensively by McClendon [5], Siegel [7], and C. A. Robinson [6]. Before proceeding to the proof of the Main Theorem we summarize their results in a form which will be convenient for our application.

Let $F \rightarrow E \xrightarrow{p} B$ be a fibration which is not necessarily simple, that is, $\pi_1(B)$ may act nontrivially on $\pi_*(F)$. We will assume throughout that F, E, B are all path connected and that F is simply connected. In this way we get a natural action of $\pi_1(B)$ on the homotopy of F . Let G be an abelian group and $h: \pi_1(B) \rightarrow \text{Aut}(G)$ a homomorphism determining up to isomorphism a local coefficient system G_h on B . Define a map $\tilde{h}: \pi_1(B) \rightarrow \text{Aut}(H^q(F; G))$ in the following way. Given an element b in $\pi_1(B)$, b determines an automorphism $\bar{b}: F \rightarrow F$ in the usual way. Let

$$\bar{b}^*: H^q(F; G) \rightarrow H^q(F; G)$$

be the induced isomorphism in cohomology. The element b also determines an isomorphism $h(b): G \rightarrow G$ which induces a coefficient isomorphism

$$h(b)_*: H^q(F; G) \rightarrow H^q(F; G).$$

Now define $\tilde{h}(b)$ to be the composite $h(b)_* \circ \bar{b}^*$. The map \tilde{h} determines a new local system on B denoted $H^q(F; G)_{\tilde{h}}$. We now have the following "twisted Serre spectral sequence".

THEOREM 1 (SIEGEL [7]). *There exists a cohomology spectral sequence $E_r^{p,q}$ such that*

$$E_2^{p,q} = H^p(B; H^q(F; G)_{\tilde{h}}), \quad E_r^{p,q} \rightarrow H^*(E; p^{-1}G_h).$$

REMARKS. $p^{-1}G_h$ denotes the local coefficient system on E induced by the map $p: E \rightarrow B$. McClendon [5] has given a relative version of this theorem.

From this spectral sequence one obtains the following results.

COROLLARY 1. *Let $H^p(B; H^q(F; G)_{\tilde{h}}) = 0$ whenever $0 < q < n$ or $0 < p < m$. Then there is an exact sequence*

$$\begin{aligned} \dots \rightarrow H^k(B; G_h) \xrightarrow{p^*} H^k(E; p^{-1}G_h) \xrightarrow{i^*} [H^k(F; G)]^{\tilde{h}} \\ \xrightarrow{\tau} H^{k+1}(B; G_h) \rightarrow \dots \rightarrow H^{m+n-1}(E; p^{-1}G_h). \end{aligned}$$

The group

$$[H^k(F; G)]^{\tilde{h}} = \{a \in H^k(F; G) \mid \tilde{h}(a) = a \text{ for all } a \text{ in } \pi_1(B)\},$$

that is, the subgroup of elements invariant under the action \tilde{h} .

COROLLARY 2. *Assume $H^p(B; H^q(F; G)_{\tilde{h}}) = 0$ for $0 < q < n$. Then*

$$p^*: H^i(B; G_h) \rightarrow H^i(E; p^{-1}G_h)$$

is an isomorphism for $i < n$, and the following sequence is exact.

$$\begin{aligned}
 0 \rightarrow H^n(B; G_h) \xrightarrow{p^*} H^n(E; p^{-1}G_h) \xrightarrow{i^*} [H^n(F; G)]^{\hat{h}} \\
 \xrightarrow{\tau} H^{n+1}(B; G_h) \xrightarrow{p^*} H^{n+1}(E; p^{-1}G_h).
 \end{aligned}$$

These sequences will be important later on when we construct the Postnikov tower for the map p .

Next we consider classifying spaces for twisted cohomology. We will confine our attention to twisted integer cohomology, since this will suffice for our applications. A much more general treatment, however, is possible. Let $h: Z_2 \rightarrow \text{Aut}(Z)$ be the nontrivial action of Z_2 on Z , and let $Z_2 \rightarrow \tilde{K} \rightarrow K(Z_2, 1)$ be the universal cover. A space of type $K(Z, n)$ can be constructed so that h determines an action \hat{h} of Z_2 on $K(Z, n)$ by basepoint preserving homeomorphisms. Take the Cartesian product $\tilde{K} \times K(Z, n)$, and let Z_2 act by covering transformations on the first factor and by \hat{h} on the second. Denote the quotient space of this action by $L(Z, n)$. Clearly, one has a fibration

$$K(Z, n) \xrightarrow{i} L(Z, n) \xrightarrow{p} K(Z_2, 1).$$

Furthermore, since the action of Z_2 on $K(Z, n)$ has a fixed point, the fibration has a canonical section $s: K(Z_2, 1) \rightarrow L(Z, n)$.

Let X be a complex and $w: X \rightarrow K(Z_2, 1)$ a fixed map. The map w determines a local system of integers on X , denoted Z_w . Let $[X, L(Z, n); w]$ denote the set of homotopy classes of lifts of w into $L(Z, n)$, where the homotopies are required to go through lifts of w . Then one has the following result.

THEOREM 2. *There is an element e_n in $H^n(L(Z, n); Z_p)$ with the property that the mapping*

$$\begin{array}{ccc}
 [X, L(Z, n); w] & \rightarrow & H^n(X; Z_w) \\
 f & \rightarrow & f^*(e_n)
 \end{array}$$

is a bijection.

The proof of this theorem can be found in Steenrod [8, 37.5]. It is not difficult to show that $s^*(e_n) = 0$ and $i^*(e_n) = \iota_n$, where ι_n in $H^n(K(Z, n); Z)$ is the fundamental class.

Let $k_0 \in K(Z, n)$ be the basepoint, and let $PK(Z, n)$ be the space of paths in $K(Z, n)$ starting at k_0 . Since the action \hat{h} of Z_2 on $K(Z, n)$ fixes the basepoint, Z_2 acts on $PK(Z, n)$ in the obvious way. As before, we can form the space $\tilde{K} \times PK(Z, n)$ and take the quotient of the Z_2 action. The resulting space will be denoted $\bar{P}L(Z, n)$ and is the total space of a fibration:

$$PK(Z, n) \rightarrow \bar{P}L(Z, n) \xrightarrow{\bar{p}} K(Z_2, 1).$$

Further, let $1 \times e: \tilde{K} \times PK(Z, n) \rightarrow \tilde{K} \times K(Z, n)$ be defined by $(1 \times e)(\tilde{k}, a) = (\tilde{k}, a(1))$. This map is a fibration with fibre $\Omega K(Z, n)$, and it is easily checked that $1 \times e$ is Z_2 equivariant. Thus $1 \times e$ induces a map $\bar{e}: \bar{P}L(Z, n) \rightarrow L(Z, n)$ which is

also a fibration with fibre $\Omega K(Z, n)$. We can summarize the above constructions in the following commutative diagram.

$$\begin{array}{ccccc}
 K(Z, n - 1) & \rightarrow & PK(Z, n) & \xrightarrow{e} & K(Z, n) \\
 \parallel & & \downarrow & & \downarrow \\
 K(Z, n - 1) & \rightarrow & \bar{P}L(Z, n) & \xrightarrow{\bar{e}} & L(Z, n) \\
 & & \downarrow \bar{p} & & \downarrow p \\
 & & K(Z_2, 1) & = & K(Z_2, 1)
 \end{array}$$

Note that we have used the fact that $\Omega K(Z, n)$ has the same homotopy type as $K(Z, n - 1)$.

There is one last space we will need. Let Z_2 act on $\Omega K(Z, n)$ in the manner analogous to the above and form the space $\bar{\Omega}L(Z, n)$. As before this space fibres over $K(Z_2, 1)$ with fibre $\Omega K(Z, n)$. It is easy to show that $\bar{\Omega}L(Z, n)$ has the same homotopy type as $L(Z, n - 1)$.

Next we consider the analogue of a principal fibration in ordinary Postnikov theory. Let B be a space with a local system of integers Z_w determined by a map $w: B \rightarrow K(Z_2, 1)$. Let v be an element of $H^{n+1}(B; Z_w)$. Consider the diagram

$$\begin{array}{ccccc}
 K(Z, n) & \xrightarrow{i} & E & \xrightarrow{\bar{e}} & \bar{P}L(Z, n + 1) \\
 & & \downarrow p_1 & & \downarrow \bar{e} \\
 & & B & \xrightarrow{v} & L(Z, n + 1).
 \end{array}$$

The fibration p_1 is the pullback by v of \bar{e} . Thus

$$E = \{(b, u) \in B \times \bar{P}L(Z, n) \mid v(b) = \bar{e}(u)\}.$$

Observe that E maps to $K(Z_2, 1)$ by the composite wp_1 . This defines an induced local system of integers on E . Define the space

$$E \oplus \bar{\Omega}L(Z, n + 1) = \{((b, u), s) \in E \times \bar{\Omega}L(Z, n + 1) \mid w(b) = p(s)\},$$

where p denotes the fibration $p: \bar{\Omega}L(Z, n + 1) \rightarrow K(Z_2, 1)$. Define a map

$$(*) \quad \mu: E \oplus \bar{\Omega}L(Z, n + 1) \rightarrow E \quad \text{by } \mu((b, u), s) = (b, s + u).$$

This map is the analogue of the principal action map for ordinary principal fibrations in the following sense. Suppose X is a complex, and $q: X \rightarrow B$ is a given map. Let $f, g: X \rightarrow E$ be two lifts of q , so that $p_1 f \simeq p_1 g \simeq q$. Define the space

$$X \oplus X = \{(x, x') \in X \times X \mid wq(x) = wq(x')\}.$$

PROPOSITION. *There exists a map $d: X \rightarrow \bar{\Omega}L(Z, n + 1) \cong L(Z, n)$ such that the composite*

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus d} E \oplus \bar{\Omega}L(Z, n + 1) \xrightarrow{\mu} E$$

is homotopic to g . Moreover d satisfies the property that $pd = wq$, that is, the diagram

$$\begin{array}{ccc} X & \xrightarrow{d} & \bar{\Omega}L(Z, n + 1) \\ wq \searrow & & \swarrow p \\ & & K(Z_2, 1) \end{array}$$

commutes. Thus, by the identification

$$\bar{\Omega}L(Z, n + 1) \cong L(Z, n),$$

we have $d \in H^n(X; q^{-1}Z_w)$.

COROLLARY. *If d represents the zero element of $H^n(X; q^{-1}Z_w)$, then f is homotopic to g .*

The map $\mu: E \oplus \bar{\Omega}L(Z, n + 1) \rightarrow E$ also has an inverse T defined by $T(b, s) = ((b, s), c)$ where $b \in B$, $s \in \bar{P}L(Z, n)$, and c is the constant path in $K(Z, n + 1)$ over the point $w(b)$. It is not difficult to see that $\mu T \simeq 1_E$.

Having given a cursory description of the construction and properties of the spaces required for nonsimple Postnikov theory, we are in a position to describe the actual tower for the fibration $F \rightarrow E \xrightarrow{p} B$. To do this we need the following result.

THEOREM 3 (McCLENDON [5]). *Assume F is $(n - 1)$ connected for $n \geq 2$. Let ι in $H^n(F; \pi_n(F))$ be the fundamental class and $h: \pi_1(B) \rightarrow \text{Aut}(\pi_n(F))$ the local system defined by the fibration. Then ι is in $[H^n(F; \pi_n(F))]^h$, and, hence, by Corollary 2 to Theorem 1, it is transgressive.*

The Postnikov tower for $F \rightarrow E \rightarrow B$ can now be constructed. Let ι in $H^n(F; \pi_n(F))$ be the fundamental class of F , and let $h: \pi_1(B) \rightarrow \text{Aut}(\pi_n(F))$ be the local system on B defined by the fibration. By the previous theorem, $\tau(\iota) = k$, is defined and is an element of $H^{n+1}(B; \pi_n(F)_h)$. Construct the diagram

$$\begin{array}{ccccc} k(Z, n) & \xrightarrow{\quad} & E_1 & & \\ & \searrow q_1 & \downarrow p_1 & & \\ E & \xrightarrow{\quad p \quad} & B & \xrightarrow{k} & L(\pi_n(F), n + 1) \end{array}$$

E_1 is the total space of the fibration which is the pullback by k of the fibration $\bar{P}L(\pi_n(F), n + 1) \rightarrow L(\pi_n(F), n + 1)$. Since $\tau(\iota) = k$, by exactness we must have $p^*(k) = 0$. Therefore, there exists a map $q_1: E \rightarrow E_1$ such that $p_1 q_1 = p$. By a standard argument, the fibre F_1 of q_1 is n -connected and $\pi_k(F_1) = \pi_k(F)$ for $k > n$. The space E_1 is the second stage of the Postnikov tower. To construct higher stages the above construction is iterated on the successive fibrations q_i .

Proof of the Main Theorem. We begin by constructing a Postnikov tower for the fibration

$$V_{m,2} \rightarrow BO(m - 2) \xrightarrow{p} BO(m)$$

where m is even, and $V_{m,2}$ is the Stiefel manifold of 2-frames in euclidean m space. The fibre $V_{m,2}$, hereafter denoted simply by V , is $(m - 3)$ connected; $\pi_{m-2}(V) = \mathbb{Z}$,

and $\pi_{m-1}(V) = Z \oplus Z_2$. The action of $\pi_1 BO(n) = Z_2$ is nontrivial on $\pi_{m-2}(V)$ and on the Z -summand of $\pi_{m-1}(V)$.

LEMMA 1. Let ι in $H^{m-2}(V; Z)$ be the fundamental class of V . Then $\tau(\iota) = \mathfrak{O}_{m-1}^{\mathbb{S}}$ in $H^{m-1}(BO(m); Z_{w_1})$.

LEMMA 2. Let $\sigma: H^m(BO(m); Z_{w_1}) \rightarrow H^{m-1}(V; Z)$ denote the suspension map for the fibration p . Then $\sigma(\chi)$ contains a spherical class.

REMARK. For the definition and properties of spherical classes the reader is referred to the article of McClendon [5].

These two lemmas imply that we may use $\mathfrak{O}_{m-1}^{\mathbb{S}}$ and χ in the first stage of our Postnikov tower to kill off $\pi_{m-2}(V)$ and the Z -summand of $\pi_{m-1}(V)$. Thus we have the diagram

$$\begin{array}{ccccc}
 & & E_1 & \xrightarrow{k} & K(Z_2, m) \\
 & \nearrow q_1 & \downarrow p_1 & & \\
 BO(m-2) & \xrightarrow{p} & BO(m) & \xrightarrow{(\mathfrak{O}_{m-1}^{\mathbb{S}}, \chi)} & L(Z, m-1) \oplus L(Z, m)
 \end{array}$$

$L(Z, m-1) \oplus L(Z, m)$ denotes the fibre product over $K(Z_2, 1)$ of $L(Z, m)$ and $L(Z, m-1)$. The map p lifts to a map $q_1: BO(m-2) \rightarrow E_1$, since $p^*(\chi)$ and $p^*(\mathfrak{O}_{m-1}^{\mathbb{S}})$ are both zero. Convert the lift q_1 into a fibration with fibre F_1 . Then F_1 is $(m-2)$ connected, and $\pi_{m-1}(F_1) = Z_2$. Let $u \in H^{m-1}(F_1; Z_2)$ be the fundamental class, and set $k = \tau(u)$ in $H^m(E_1; Z_2)$. The element k is the second Postnikov invariant for our fibration p . Let $K(Z_2, m-1) \rightarrow E_2 \xrightarrow{p_2} E_1$ be the principal fibration induced by k . We then have the following diagram.

$$\begin{array}{ccccc}
 & & E_2 & & \\
 & \nearrow q_2 & \downarrow p_2 & & \\
 & \nearrow q_1 & E_1 & \xrightarrow{k} & K(Z_2, m) \\
 & \nearrow p & \downarrow p_1 & & \\
 BO(m-2) & \xrightarrow{p} & BO(m) & \xrightarrow{(\mathfrak{O}_{m-1}^{\mathbb{S}}, \chi)} & L(Z, m-1) \oplus L(Z, m)
 \end{array}$$

The fibre F_2 of q_2 is $(m-1)$ connected, and therefore by a standard argument the map $q_{2\sharp}: [X, BO(m-2)] \rightarrow [X, E_2]$ is a bijection for a complex X of dimension less than or equal to m . Thus if we are given a map $f: X \rightarrow BO(m)$, f lifts to $g: X \rightarrow BO(m-2)$ if, and only if, f lifts to E_2 . If $f^*(\chi) = f^*(\mathfrak{O}_{m-1}^{\mathbb{S}}) = 0$, then f lifts to a map $g: X \rightarrow E_1$, and it is clear that f lifts to E_2 if, and only if, we can choose a lift $g: X \rightarrow E_1$ for which $g^*(k) = 0$.

LEMMA 3. Define \bar{E} to be the space $E_1 \oplus L(Z, m - 2) \oplus L(Z, m - 1)$, the fibre product over $K(Z_2, 1)$. Let $\mu: \bar{E} \rightarrow E_1$ be the map defined by $(*)$, and let $e_{m-2} \in H^{m-2}(L(Z, m - 2); Z_p)$ and $e_{m-1} \in H^{m-1}(L(Z, m - 1); Z_p)$ be the twisted fundamental classes of Theorem 2. Then we have the following formula.

$$\begin{aligned} \mu^*(k) &= k \otimes 1 \otimes 1 + 1 \otimes \text{Sq}^1 e_{m-1} \otimes 1 + 1 \otimes 1 \\ &\quad \otimes \text{Sq}^2 e_{m-2} + w_2 \otimes 1 \otimes e_{m-2}. \end{aligned}$$

The proofs of Lemmas 1-3 will be given later.

COROLLARY. If g_1 and g_2 are two lifts of f to E_1 , then there exist elements u_1 in $H^{m-1}(X; Z_{f^*(w_1)})$ and $u_2 \in H^{m-2}(X; Z_{f^*(w_1)})$ such that

$$g_2^*(k) = g_1^*(k) + \text{Sq}^1 u_1 + \text{Sq}^2 u_2 + f^*(w_2) u_2.$$

PROOF. Use Lemma 3 and the Proposition.

The corollary shows that the elements $g^*(k)$ as g runs over lifts of f to E_1 form a coset in $H^m(X; Z_2)$ of the subgroup

$$\text{Sq}^1 H^{m-1}(X; Z_{f^*(w_1)}) + (\text{Sq}^2 + f^*(w_2)) H^{m-2}(X; Z_{f^*(w_1)}).$$

This immediately gives us the following theorem.

THEOREM 4. Let X be a complex of dimension m , where m is even. Suppose $f: X \rightarrow BO(m)$ satisfies $f^*(\mathcal{O}_{m-1}^{\otimes 2}) = 0$ and $f^*(\chi) = 0$. Then f lifts to $f: X \rightarrow BO(m - 2)$ if, and only if, $g^*(k) = 0$ in

$$H^m(X; Z_2) / \text{Sq}^1 H^{m-1}(X; Z_{f^*(w_1)}) + (\text{Sq}^2 + f^*(w_2)) (H^{m-2}(X; Z_{f^*(w_1)}))$$

where g is any lift of f to E_1 .

The Main Theorem is now an easy consequence of the following lemma.

LEMMA 4. Let M be a nonorientable manifold of dimension m . The map $\text{Sq}^1: H^{m-1}(M; Z_{w_1(M)}) \rightarrow H^m(M; Z_2)$ is onto.

PROOF. We have the following commutative triangle:

$$\begin{array}{ccc} H^{m-1}(M; Z_{w_1(M)}) & \xrightarrow{\text{Sq}^1} & H^m(M; Z_2) \\ & \searrow \rho_2 & \nearrow \text{Sq}^1 \\ & & H^{m-1}(M; Z_2) \end{array}$$

where ρ_2 is reduction mod 2. It is well known that $\text{Sq}^1: H^{m-1}(M; Z_2) \rightarrow H^m(M; Z_2)$ is onto. But ρ_2 is also onto as can be seen from the following exact sequence:

$$\dots \rightarrow H^{m-1}(M; Z_{w_1(M)}) \xrightarrow{\rho_2} H^{m-1}(M; Z_2) \xrightarrow{\beta} H^m(M; Z_{w_1(M)}) \rightarrow \dots,$$

where β is the Bockstein homomorphism associated to the coefficient sequence $0 \rightarrow Z_{w_1} \rightarrow Z_{w_1} \rightarrow Z_2 \rightarrow 0$. Since $H^m(M; Z_{w_1(M)})$ is infinite cyclic and $H^{m-1}(M; Z_2)$

has order 2, we immediately see that β is zero, and hence ρ_2 is onto. Thus $\text{Sq}^1: H^{m-1}(M; Z_{w_1(M)}) \rightarrow H^m(M; Z_2)$ is the composition of onto maps and must itself be onto.

To prove the Main Theorem let $f: M \rightarrow BO(m)$ be a map classifying $T(M)$, the tangent bundle of M . By hypothesis, $f^*(\chi)$ and $f^*(\mathcal{O}^1_{m-1})$ are zero. Choose any lift of f to the space E_1 ; call it g . Then $g^*(k)$ is an element of the quotient group

$$H^m(M; Z_2) / \text{Sq}^1 H^{m-1}(M; Z_{w_1(M)}) + (\text{Sq}^2 + w_2(M)) H^{m-2}(M; Z_{w_1(M)}).$$

But this group is zero by Lemma 4, and so by Theorem 4 f lifts to $BO(m - 2)$.

It only remains for us to prove Lemmas 1 through 3.

PROOF OF LEMMA 1. Corollary 2 to Theorem 1 gives us the following exact sequence.

$$\rightarrow [H^{m-2}(V; Z)]^{\bar{h}} \xrightarrow{\tau} H^{m-1}(BO(m), Z_{w_1}) \xrightarrow{p^*} H^{m-1}(BO(m-2); Z_{w_1}) \rightarrow .$$

The group on the left is Z , generated by ι according to Theorem 3. The kernel of p^* is generated by \mathcal{O}^1_{m-1} , so by exactness $\tau(\iota) = \mathcal{O}^1_{m-1}$.

PROOF OF LEMMA 2. Consider the diagram

$$\begin{array}{ccccc} V_{m,2} & \rightarrow & BO(m-2) & \xrightarrow{p} & BO(m) \\ g \downarrow & & \downarrow & & \parallel \\ S^{m-1} & \rightarrow & BO(m-1) & \xrightarrow{p'} & BO(m) \end{array}$$

Let σ be the suspension for p and σ' the suspension for p' . By the Serre and Gysin sequences for p' it is easy to see that $\sigma'(\chi) = v$, where v is a generator of $H^{m-1}(S^{m-1}; Z)$. So $\sigma'(\chi)$ contains a spherical class, namely v . The map g is a fibration with fibre S^{m-2} . Consider the homotopy exact sequence for g :

$$\rightarrow \pi_{m-1}(V_{m,2}) \xrightarrow{g_*} \pi_{m-1}(S^{m-1}) \xrightarrow{\Delta} \pi_{m-2}(S^{m-2}) \xrightarrow{i_*} \pi_{m-2}(V_{m,2}) \xrightarrow{g_*} \pi_{m-2}(S^{m-1}).$$

Since $\pi_{m-2}(S^{m-1})$ is zero, we must have i_* onto. But $\pi_{m-2}(S^{m-2})$ and $\pi_{m-2}(V_{m,2})$ are both infinite cyclic, and consequently i_* is also 1-1. By exactness, this makes $g_*: \pi_{m-1}(V_{m,2}) \rightarrow \pi_{m-1}(S^{m-1})$ an epimorphism. This implies that $g^*(v)$ in $\sigma(\chi)$ is spherical.

PROOF OF LEMMA 3. Consider the following diagram:

$$\begin{array}{ccc} BO(m-2) \oplus L(Z, m-2) \oplus L(Z, m-1) & \xrightarrow{q_1 \oplus 1 \oplus 1} & \bar{E} \xrightarrow{\mu} E_1 \\ \bar{s} \uparrow \downarrow \pi & & \downarrow p_1 \\ BO(m-2) & \xrightarrow{p} & BO(m) \end{array}$$

Recall that $\bar{E} = E_1 \oplus L(Z, m-2) \oplus L(Z, m-1)$, the fibre product over $K(Z_2, 1)$. It is easy to see that the map π has a section \bar{s} . Furthermore, if we let v be the

composite $\mu \circ (q_1 \oplus 1 \oplus 1)$, then $v\bar{s} \simeq q_1$. By the work of Emery Thomas [9], there exist a map τ_1 and a short exact sequence in Z_2 cohomology

$$0 \rightarrow H^m(E_1) \xrightarrow{v^*} H^m(BO(m-2) \oplus L(Z, m-2) \oplus L(Z, m-1)) \xrightarrow{\tau_1} H^{m+1}(BO(m)).$$

Moreover, τ_1 commutes with Steenrod squares and is an $H^*(BO(n); Z_2)$ module map. As described earlier, there is a map $T: E_1 \rightarrow \bar{E}$ such that $\mu T \simeq 1_{\bar{E}}$. Therefore,

$$\mu^*(k) = k \otimes 1 \otimes 1 + \sum_j a_j \otimes b_j \otimes c_j$$

where a_j is in $H^*(E_1; Z_2)$, b_j is in $H^*(L(Z, m-2); Z_2)$, and c_j is in

$$H^*(L(Z, m-1); Z_2),$$

with $\deg a_j + \deg b_j + \deg c_j = m$ and $\deg b_j + \deg c_j > 0$. This last condition implies that $\sum a_j \otimes b_j \otimes c_j$ is in $\ker \bar{s}^*$. The second Postnikov invariant k is in $\ker q_1^*$, so $v^*(k) = \sum a_j \otimes b_j \otimes c_j$ is in $\ker \bar{s}^* \cap \ker \tau_1$ by the short exact sequence above.

LEMMA 5. *The intersection of $\ker \bar{s}^*$ with $H^m(BO(m-2) \oplus (L(Z, m-1); Z_2))$ consists of the elements $1 \otimes \text{Sq}^1 e_{m-1} \otimes 1$, $1 \otimes 1 \otimes \text{Sq}^2 e_{m-2}$, $w_1 \otimes 1 \otimes \text{Sq}^1 e_{m-2}$, and $w_2 \otimes 1 \otimes e_{m-2}$.*

PROOF. Consider the mod 2 Serre spectral sequence for the fibration π (see previous diagram). Since π has a section, and since we are taking mod 2 cohomology, an easy argument shows that the spectral sequence collapses, that is, $E_2 = E_\infty$. The conclusion of the lemma now follows easily.

We now compute the action of the map τ_1 on these elements. Recall that by construction we have $\tau_1(1 \otimes e_{m-1} \otimes 1) = w_m$, where w_m is the mod 2 reduction of χ , and $\tau_1(1 \otimes 1 \otimes e_{m-2}) = w_{m-1}$. Using these formulas and those of Wu [11] we obtain

$$\begin{aligned} \tau_1(1 \otimes \text{Sq}^1 e_{m-1} \otimes 1) &= \text{Sq}^1 w_m = w_1 w_m, \\ \tau_1(1 \otimes 1 \otimes \text{Sq}^2 e_{m-2}) &= \text{Sq}^2 w_{m-1} = w_2 w_{m-1} + w_1 w_m, \\ \tau_1(w_1 \otimes 1 \otimes \text{Sq}^1 e_{m-2}) &= w_1^2 w_{m-1}, \\ \tau_1(w_2 \otimes 1 \otimes e_{m-2}) &= w_2 w_{m-1}. \end{aligned}$$

Therefore, $\ker \tau_1 \cap \ker \bar{s}^*$ must consist of the single element

$$1 \otimes \text{Sq}^1 e_{m-1} \otimes 1 + 1 \otimes 1 \otimes \text{Sq}^2 e_{m-2} + w_2 \otimes 1 \otimes e_{m-2}.$$

By the short exact sequence given earlier, this gives us the required formula for $\mu^*(k)$.

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